§1.1 Systems of Linear Equations

Given variables $x_1, \ldots, x_n$, a linear equation is an expression

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = b$$

where b and the coefficients $c_1, \ldots, c_n$ are real or complex numbers.

For example:

- $4x_1 + \sqrt{5}x_2 - 7 = 2x_3 \Longrightarrow 4x_1 + \sqrt{5}x_2 - 2x_3 = 7$
- $3x_1 = 7x_2 + 4 \Longrightarrow 3x_1 - 7x_2 = 4$

are linear equations since they can be rearranged to that format.

However, expressions like

$$x_1^2 + x_2^2 = 1, \quad x_1 + x_2 x_3 = 9, \quad x_1 x_2 x_3 = 1$$

are NOT linear equations.

A collection of linear equations is called a system of linear equations (linear system)

For example,

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 8 \\ 3x_1 \qquad\quad + 12x_3 = 16 \end{cases}$$

is a linear system.

Given a system of linear equations in n variables $x_1, \ldots, x_n$, we say that $(s_1, \ldots, s_n)$ is a <u>solution</u> of the system if it satisfies each equation in the system.

(easy) Exercise

Verify that $\left(\frac{4}{3}, \frac{4}{3}, 1\right)$ is a solution to the system above.

Given a linear system, the set of all possible solutions is called the <u>solution set</u>.

Remark

The solution set of a linear system consists of either

1) No solutions

2) One unique solution

3) Infinitely many solutions

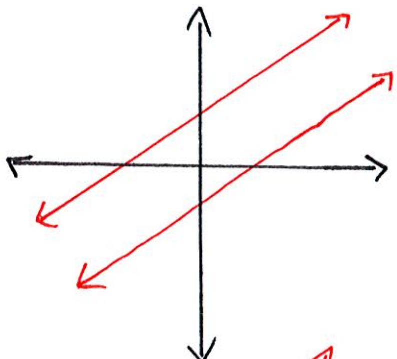
Some geometric intuition for the remark.

Let

$$\begin{cases} a_1 x_1 + a_2 x_2 = c \\ b_1 x_1 + b_2 x_2 = d \end{cases}$$

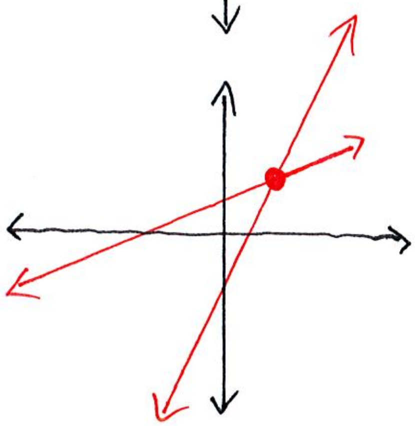
be a linear system in variables x_1, x_2. Any solution (s_1, s_2) is a point of intersection of the two lines defined by these equations. With this there are 3 possibilities

1)



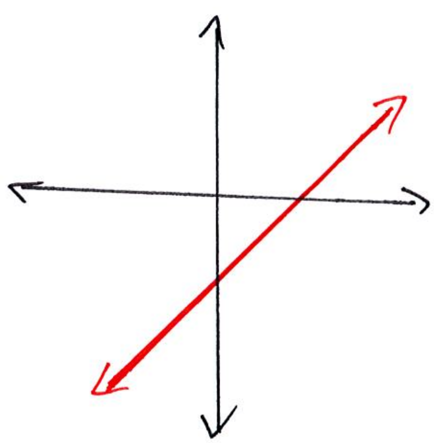
parallel lines. No intersections

2)



Lines intersect at exactly 1 point.

3)



The 2 equations define the same line. Then they intersect at every point on the line.

We say a linear system is consistent if it has either one solution or infinitely many solutions.

We say a linear system is inconsistent if it has no solution.

Goal: Determine the solution sets of systems of linear equations.

To do this we use matrix notation.

Example

Given the system

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 5x_2 = 1 \\ x_2 + 2x_3 = 9 \end{cases}$$

we can associate the coefficient matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

and the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \\ 0 & 1 & 2 & 9 \end{array}\right]$$

Book and MyLab don't draw this line, but we do